RANK-CRANK TYPE PDES AND GENERALIZED LAMBERT SERIES IDENTITIES

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Dedicated to our friends, Mourad Ismail and Dennis Stanton

Abstract. We show how Rank-Crank type PDEs for higher order Appell functions due to Zwegers may be obtained from a generalized Lambert series identity due to the first author. Special cases are the Rank-Crank PDE due to Atkin and the third author and a PDE for a level 5 Appell function also found by the third author. These two special PDEs are related to generalized Lambert series identities due to Watson, and Jackson respectively. The first author's Lambert series identities are common generalizations. We also show how Atkin and Swinnerton-Dyer's proof using elliptic functions can be extended to prove these generalized Lambert series identities.

1. Introduction

F. J. Dyson [9], [10, p. 52] defined the rank of a partition as the largest part minus the number of parts. Dyson conjectured that the residue of the rank mod 5 divides the partitions of $5n+4$ into 5 equal clases thereby providing a combinatorial interpretation of Ramanujan's famous partition congruences $p(5n+4) \equiv 0 \pmod 5$. He also conjectured that the rank mod 7 likewise gives Ramanujan's partition congruence $p(7n+5) \equiv 0 \pmod 7$. Dyson's rank conjectures were proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [3]. The following was the crucial identity that Atkin and Swinnerton-Dyer needed for the proof of the Dyson rank conjectures. It was first proved by G.N. Watson [18].

$$\zeta \frac{[\zeta^2]_\infty}{[\zeta]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n} + \frac{[\zeta]_\infty [\zeta^2]_\infty (q)_\infty^2}{[z/\zeta]_\infty [z]_\infty [\zeta z]_\infty} \tag{1.1}$$
$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n+1)/2} \left(\frac{\zeta^{-3n}}{1-zq^n/\zeta} + \frac{\zeta^{3n+3}}{1-z\zeta q^n} \right).$$

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Throughout we use the standard q-notation

$$\begin{aligned}
(x)_0 &:= (x;q)_0 := 1,\\
(x)_n &:= (x;q)_n := \prod_{m=0}^{n-1}(1-xq^m),\\
(x_1,\cdots,x_m)_n &:= (x_1,\cdots,x_m;q)_n := (x_1;q)_n\cdots(x_m;q)_n,\\
[x_1,\cdots,x_m]_n &:= [x_1,\cdots,x_m;q]_n := (x_1,q/x_1,\cdots,x_m,q/x_m;q)_n.
\end{aligned}$$

when n is a nonnegative integer. Assuming $|q|<1$ we also use this notation when $n=\infty$ by interpreting its meaning as the limit as $n\to\infty$. Later M. Jackson [14] proved an analogue of the above identity,

$$\begin{aligned}
&\frac{\zeta^2[\zeta^2]_\infty[x\zeta]_\infty[x/\zeta]_\infty}{[\zeta]_\infty[x]_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n(n+1)/2}}{1-zq^n}+\frac{[\zeta]_\infty[\zeta^2]_\infty[x\zeta]_\infty[\zeta/x]_\infty(q)_\infty^2}{[z/x]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[zx]_\infty}\\
&+\frac{\zeta}{x}\frac{[\zeta]_\infty[\zeta^2]_\infty}{[x]_\infty[x^2]_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{5n(n+1)/2}\left(\frac{x^{-5n}}{1-zq^n/x}+\frac{x^{5n+5}}{1-zxq^n}\right) \qquad (1.2)\\
&=\sum_{n=-\infty}^{\infty}(-1)^nq^{5n(n+1)/2}\left(\frac{\zeta^{-5n}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{5n+5}}{1-z\zeta q^n}\right).
\end{aligned}$$

Recently, the first author [8, p.603] found a generalization of the above two identities, namely,

$$\begin{aligned}
&\frac{x_1^m[x_2/x_1,\cdots,x_m/x_1,x_1x_m,\cdots,x_1x_2,x_1^2]_\infty}{[x_1]_\infty[x_2,\cdots,x_m]_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(2m+1)n(n+1)/2}}{1-zq^n}\\
&+\frac{[x_1/x_2,\cdots,x_1/x_m,x_1,x_1x_m,\cdots,x_1x_2,x_1^2]_\infty(q)_\infty^2}{[z/x_1,z/x_2,\cdots,z/x_m,z,zx_m,\cdots,zx_1]_\infty}\\
&+\Bigg\{\frac{x_1}{x_2}\frac{[x_1/x_3,\cdots,x_1/x_m,x_1,x_1x_m,\cdots,x_1x_3,x_1^2]_\infty}{[x_2/x_3,\cdots,x_2/x_m,x_2,x_2x_m,\cdots,x_2^2]_\infty} \qquad (1.3)\\
&\quad\times\sum_{n=-\infty}^{\infty}(-1)^nq^{(2m+1)n(n+1)/2}\left(\frac{x_2^{-(2m+1)n}}{1-zq^n/x_2}+\frac{x_2^{(2m+1)(n+1)}}{1-zx_2q^n}\right)+\mathrm{idem}(x_2;x_3,\cdots,x_m)\Bigg\}\\
&=\sum_{n=-\infty}^{\infty}(-1)^nq^{(2m+1)n(n+1)/2}\left(\frac{x_1^{-(2m+1)n}}{1-zq^n/x_1}+\frac{x_1^{(2m+1)(n+1)}}{1-zx_1q^n}\right),
\end{aligned}$$

where $g(a_1,a_2,\cdots,a_m)+$ idem$(a_1;a_2,\cdots,a_n)$ denotes the sum

$$\sum_{i=1}^{n}g(a_i,a_2,\cdots,a_{i-1},a_1,a_{i+1},\cdots,a_m),$$

in which the i-th term of the sum is obtained from the first by interchanging a_1 and a_i.

Equation (1.3) was proved using partial fractions. Indeed, the $m=1$ case of (1.3) is equivalent to (1.1), while the $m=2$ case is equivalent to (1.2). The fact that the right-hand side of (1.2) is independent of x, and that the right-hand side of (1.3) is

independent of $x_2, x_3, \cdots, x_m$ seems to be intriguing at first. Indeed, one purpose of this article is to show that the left-hand sides of (1.2) and (1.3) are really elliptic functions of order less than 2, in fact entire functions as we show, in the respective variables (x for (1.2) and x_2 for (1.3) while holding $x_3, \cdots, x_m$ fixed) and therefore that they must be constants which are nothing but the right-hand sides of (1.2) and (1.3) respectively. Since (1.2) follows from (1.3), we show this only for (1.3). This is done in Section 2.

Let $N(m,n)$ denote the number of partitions of n with rank m. Then the rank generating function $R(z,q)$ is given by

$$R(z,q) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}. \tag{1.4}$$

In [1], G. E. Andrews and the third author defined the crank of a partition, a partition statistic hypothesized by Dyson in [9]. It is the largest part if the partition contains no ones, and otherwise is the number of parts larger than the number of ones minus the number of ones. For $n > 1$, we let $M(m,n)$ denote the number of partitions of n with crank m. If we amend the definition of $M(m,n)$ for $n = 1$, then the generating function can be given as an infinite product. Accordingly, we assume

$$M(0,1) = -1,\ M(-1,1) = M(1,1) = 1, \text{ and } M(m,1) = 0 \text{ otherwise}.$$

Then the crank generating function $C(z,q)$ is given by

$$C(z,q) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} M(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}. \tag{1.5}$$

Atkin and the third author [2] found the so-called Rank-Crank PDE, a partial differential equation (PDE) which relates $R(z,q)$ and $C(z,q)$. To state this PDE in its original form, we first define the differential operators

$$\delta_z = z\frac{\partial}{\partial z}, \qquad \delta_q = q\frac{\partial}{\partial q}. \tag{1.6}$$

Then the Rank-Crank PDE can be written as

$$z(q)_\infty^2 [C^*(z,q)]^3 = \left(3\delta_q + \frac{1}{2}\delta_z + \frac{1}{2}\delta_z^2\right) R^*(z,q), \tag{1.7}$$

where

$$\begin{aligned} R^*(z,q) &:= \frac{R(z,q)}{1-z}, \\ C^*(z,q) &:= \frac{C(z,q)}{1-z}. \end{aligned} \tag{1.8}$$

In [2], it was shown how the Rank-Crank PDE and certain results for the derivatives of Eisenstein series lead to exact relations between rank and crank moments. As in [13], define $N_k(m,n)$ by

$$\sum_{n\geq 0} N_k(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n((2k-1)n-1)/2+|m|n} (1-q^n), \tag{1.9}$$

of any positive integer k. When $k = 1$ this is the generating function for the crank, and when $k = 2$ it is the generating function for the rank. When $k \geq 2$, $N_k(m, n)$ can be interpreted combinatorially as the number of partitions of n into $k-1$ successive Durfee squares with k-rank equal to m. See [13, Eq.(1.11)] for a definition of the k-rank. We define

$$R_k(z,q) := \sum_{n\geq 0}\sum_{m=-n}^{n} N_k(m,n)z^m q^n. \tag{1.10}$$

From [13, Eq.(4.5)], this generating function can be written as

$$R_k(z,q) = \sum_{n_{k-1}\geq n_{k-2}\geq\cdots\geq n_1\geq 1} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}}, \tag{1.11}$$

when $k \geq 2$. In Section 3, we show that $R_k(z,q)$ is related to the level $2k-1$ Appell function

$$\Sigma^{(2k-1)}(z,q) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2k-1)n(n+1)/2}}{1-zq^n}. \tag{1.12}$$

We obtain the following

Theorem 1.1. *For $k \geq 1$,*

$$\begin{aligned} &R_k(z,q)\\ &= \frac{1}{(q)_\infty}\left(z^{k-1}(1-z)\Sigma^{(2k-1)}(z,q) - z\theta_{1,2k-1}(q) + z(1-z)\sum_{m=0}^{k-3} z^m\theta_{2m+3,2k-1}(q)\right), \end{aligned} \tag{1.13}$$

where

$$\theta_{j,2k-1}(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n((2k-1)n+j)/2}, \tag{1.14}$$

for $j = 1, 3, \ldots, 2k-3$.

This theorem generalizes Lemma 7.9 in [12] which gives a relation between the rank generating function $R(z,q)$ and a level 3 Appell function. The $k = 1$ case of the theorem gives the familiar partial fraction expansion for the reciprocal of Jacobi's theta product $(z)_\infty(z^{-1}q)_\infty$ [16, p. 1], [17, p. 136], since $\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2} = 0$.

A few years ago the third author found a 4^{th} order PDE, which is an analogue of the Rank-Crank PDE and is related to the 3-rank [13]. To state this PDE we define

$$G^{(5)}(z,q) := \frac{1}{(q)_\infty^3}\Sigma^{(5)}(z,q). \tag{1.15}$$

Then

$$\begin{aligned} &24(q)_\infty^2[C^*(z,q)]^5\\ &= 24(1-10\Phi_3(q))G^{(5)}(z,q)\\ &\quad + \left(100\delta_q + 50\delta_z + 100\delta_q\delta_z + 35\delta_z^2 + 20\delta_q\delta_z^2 + 100\delta_q^2 + 10\delta_z^3 + \delta_z^4\right)G^{(5)}(z,q), \end{aligned} \tag{1.16}$$

where

$$\Phi_3(q) := \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}. \tag{1.17}$$

This PDE can be written more compactly as

$$24(q)_\infty^2 [C^*(z,q)]^5 = (\mathbf{H}_*^2 - E_4)\, G^{(5)}(z,q), \tag{1.18}$$

where $\mathbf{H}_*$ is the operator

$$\mathbf{H}_* := 5 + 10\delta_q + 5\delta_z + \delta_z^2,$$

and

$$E_4 := E_4(q) := 1 + 240\Phi_3(q),$$

is the usual Eisenstein series of weight 4. The PDE (1.16) was first conjectured by the third author and then subsequently proved and generalized by Zwegers [21]. It was also Zwegers who first observed that (1.16) could be written in a more compact form. We now describe Zwegers's generalization. Define for $l \in \mathbb{Z}_{>0}$, the level l Appell function as

$$A_l(u,v) := A_l(u,v;\tau) := z^{l/2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{ln} q^{ln(n+1)/2} w^n}{1 - zq^n}, \tag{1.19}$$

where $z = e^{2\pi i u}$, $w = e^{2\pi i v}$, $q = e^{2\pi i \tau}$, and define the modified rank and crank generating functions as follows.

$$\mathcal{R} := \mathcal{R}(u;\tau) := \frac{z^{1/2} q^{-1/24}}{1-z} R(z,q), \tag{1.20}$$

$$\mathcal{C} := \mathcal{C}(u;\tau) := \frac{z^{1/2} q^{-1/24}}{1-z} C(z,q). \tag{1.21}$$

Here and throughout we assume $\operatorname{Im}\tau > 0$ so that $|q| < 1$. Then the following theorem due to Zwegers gives for odd l, the $(l-1)^{\text{th}}$ order analogue of the Rank-Crank PDE.

Theorem 1.2 (Zwegers[21])**.** *Let $l \geq 3$ be an odd integer. Define*

$$\begin{aligned} \mathcal{H}_k &:= \frac{l}{\pi i}\frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial u^2} - \frac{l(2k-1)}{12} E_2, \\ \mathcal{H}^{[k]} &:= \mathcal{H}_{2k-1}\mathcal{H}_{2k-3}\cdots\mathcal{H}_3\mathcal{H}_1, \end{aligned} \tag{1.22}$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$ is the usual Eisenstein series of weight 2 with $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then there exist holomorphic modular forms f_j (which can be constructed explicitly), with $j = 4, 6, 8, \cdots, l-1$, on $SL_2(\mathbb{Z})$ of weight j, such that

$$\left(\mathcal{H}^{[(l-1)/2]} + \sum_{k=0}^{(l-5)/2} f_{l-2k-1} \mathcal{H}^{[k]} \right) A_l(u,0) = (l-1)!\, \eta^l \mathcal{C}^l, \tag{1.23}$$

where η is the Dedekind η-function, given by $\eta(\tau) = q^{1/24}(q)_\infty$.

Zwegers proved this theorem using the formulas and methods motivated from the theory of Jacobi forms. In contrast to this, the proof of the Rank-Crank PDE by Atkin and the third author, which corresponds to the $l = 3$ case of Zwegers's PDE, depends upon simply taking the second derivative with respect to ζ of both sides of (1.1). The main goal of this paper is to show how a generalized Rank-Crank PDE of any odd order follows from the Lambert series identity (1.3) in a similar fashion. We will obtain Zwegers's result in a different form. In our form the coefficients are quasimodular forms rather than holomorphic modular forms, but in contrast, our coefficients are given recursively. See Theorem 4.4 and Corollary 4.5.

This paper is organized as follows. In Section 2, we prove (1.3) using the theory of elliptic functions. Then in Section 3 we prove Theorem 1.1, which is the theorem that relates $R_k(z,q)$ with the level $2k-1$ Appell function $\Sigma^{(2k-1)}(z,q)$. In Section 4 we prove our main result that shows how (1.3) can be used to derive the higher order Rank-Crank-type PDEs of Zwegers.

In the light of (1.19), it should be observed that the identities (1.1) and (1.2) are really the identities involving certain combinations of level 3 and level 5 Appell functions respectively while (1.3) is an identity involving a combination of level $(2m+1)$ Appell functions. However, the analogue for level 1 Appell function which cannot be derived from (1.3) was found by R. Lewis [15, Equation 11] and is as follows.

$$\frac{[z]_\infty[\zeta^2]_\infty(q)_\infty^2}{[z\zeta]_\infty[\zeta]_\infty[z\zeta^{-1}]_\infty} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}\left(\frac{\zeta^{-n}}{1-zq^n/\zeta}+\frac{\zeta^{n+1}}{1-z\zeta q^n}\right). \tag{1.24}$$

2. General Lambert series identity through elliptic function theory

Atkin and Swinnerton-Dyer's proof of (1.1) depends in essence on the theory of elliptic functions. In this section, we show how this method of proof can be used to prove (1.3). Let

$$z = e^{2\pi i u}, x_1 = e^{2\pi i v}, x_2 = e^{2\pi i w}, \tag{2.1}$$

and let

$$x_j = e^{2\pi i a_j}, \quad j = 3, \cdots, m, \tag{2.2}$$

where $u, v, w, a_3, \ldots, a_m$ are all complex numbers. Also recall that $q = e^{2\pi i\tau}$, where Im $\tau > 0$.

Let

$$[b]_\infty =: J(a,q) =: J(a), \text{ where } b = e^{2\pi i a}, a \in \mathbb{C}. \tag{2.3}$$

Then using the Jacobi triple product identity [4, p. 10, Theorem 1.3.3], we easily find that,

$$J(a,q) = \frac{ie^{\pi i a}q^{-1/8}}{(q)_\infty}\theta(a), \tag{2.4}$$

where

$$\theta(z) = \theta(z;\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i\left(n+\frac{1}{2}\right)^2\tau+2\pi i\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}. \tag{2.5}$$

Comparing this with the classical definition of $\theta_1(z)$ [11, p. 355, Section 13.19, Equation 10], we find that upon replacing q by $q^{1/2}$ in this classical definition, $\theta(z) = -\theta_1(z)$. From [20, p. 8], we see that

$$\theta(z+1) = -\theta(z), \tag{2.6}$$
$$\theta(z+\tau) = -e^{-\pi i\tau - 2\pi iz}\theta(z), \tag{2.7}$$
$$\theta(-z) = -\theta(z), \tag{2.8}$$
$$\theta'(0;\tau) = -2\pi q^{1/8}(q)_\infty^3. \tag{2.9}$$

Using (2.6) and (2.7), we have

$$J(a+1,q) = J(a,q), \tag{2.10}$$
$$J(a+\tau,q) = -e^{-2\pi ia}J(a,q), \tag{2.11}$$
$$J(a-\tau,q) = -q^{-1}e^{2\pi ia}J(a,q), \tag{2.12}$$
$$J(-a,q) = -e^{-2\pi ia}J(a,q). \tag{2.13}$$

Using (2.3), we rephrase (1.3) as follows:

$$\begin{aligned}
&\frac{e^{2\pi imv}J(w-v)J(w+v)J(2v)}{J(v)(J(w))^2}\prod_{3\le j\le m}\frac{J(a_j-v)J(a_j+v)}{(J(a_j))^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi iu}q^n}\\
&+\frac{J(v-w)J(v)J(v+w)J(2v)(q)_\infty^2}{J(u-v)J(u-w)J(u)J(u+w)J(u+v)}\prod_{3\le j\le m}\frac{J(v-a_j)J(v+a_j)}{J(u-a_j)J(u+a_j)}\\
&+\Bigg\{e^{2\pi i(v-w)}\frac{J(v)J(2v)}{J(w)J(2w)}\prod_{3\le j\le m}\frac{J(v-a_j)J(v+a_j)}{J(w-a_j)J(w+a_j)}\\
&\quad\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\right)+\mathrm{idem}(w;a_3,\cdots,a_m)\Bigg\}\\
&=\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iv(2m+1)n}}{1-e^{2\pi i(u-v)}q^n}+\frac{e^{2\pi iv(2m+1)(n+1)}}{1-e^{2\pi i(u+v)}q^n}\right).
\end{aligned}\tag{2.14}$$

Fix $a_3,\cdots,a_m$, consider the left-hand side of (2.14) as a function of w only and denote it by $g(w)$. Let $f_1(w)$ denote the expression in line 1 of (2.14), $f_2(w)$ the expression in line 2 of (2.14) and $f_3(w)$ the expression in lines 3 and 4 of (2.14). Then, using (2.10), (2.11) and (2.12), we see

$$\begin{aligned}
f_1(w+1) &= f_1(w) = f_1(w+\tau),\\
f_2(w+1) &= f_2(w) = f_2(w+\tau),\\
f_3(w+1) &= f_3(w).
\end{aligned}\tag{2.15}$$

Another application of (2.11) and (2.12) gives

$$
\begin{aligned}
&f_3(w+\tau)\\
&=\frac{e^{2\pi i(v-w-\tau)}J(v)J(2v)}{J(w+\tau)J(2w+2\tau)}\prod_{3\le j\le m}\frac{J(v-a_j)J(v+a_j)}{J(w+\tau-a_j)J(w+\tau+a_j)}\\
&\quad\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi i(w+\tau)(2m+1)n}}{1-e^{2\pi i(u-w-\tau)}q^n}+\frac{e^{2\pi i(w+\tau)(2m+1)(n+1)}}{1-e^{2\pi i(u+w+\tau)}q^n}\right)\\
&\quad+\sum_{k=3}^{m}e^{2\pi i(v-a_k)}\frac{J(v)J(2v)J(v-w-\tau)J(v+w+\tau)}{J(a_k)J(2a_k)J(a_k-w-\tau)J(a_k+w+\tau)}\prod_{\substack{2<j<m+1\\ j\neq k}}\frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)}\\
&\qquad\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi i a_k(2m+1)n}}{1-e^{2\pi i(u-a_k)}q^n}+\frac{e^{2\pi i a_k(2m+1)(n+1)}}{1-e^{2\pi i(u+a_k)}q^n}\right)\\
&=e^{2\pi iv+4\pi imw}\frac{J(v)J(2v)}{J(w)J(2w)}\prod_{3\le j\le m}\frac{J(v-a_j)J(v+a_j)}{J(w-a_j)J(w+a_j)}\Bigg(\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)(n+1)(n+2)/2}\\
&\quad\times\frac{e^{-2\pi iw(2m+1)(n+1)}q^{-(2m+1)(n+1)}}{1-e^{2\pi i(u-w)}q^n}+\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n-1)/2}\frac{e^{2\pi iw(2m+1)n}q^{(2m+1)n}}{1-e^{2\pi i(u+w)}q^n}\Bigg)\\
&\quad+\sum_{k=3}^{m}e^{2\pi i(v-a_k)}\frac{J(v)J(2v)J(v-w)J(v+w)}{J(a_k)J(2a_k)J(a_k-w)J(a_k+w)}\prod_{\substack{2<j<m+1\\ j\neq k}}\frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)}\\
&\qquad\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi i a_k(2m+1)n}}{1-e^{2\pi i(u-a_k)}q^n}+\frac{e^{2\pi i a_k(2m+1)(n+1)}}{1-e^{2\pi i(u+a_k)}q^n}\right)\\
&=f_3(w).
\end{aligned}\tag{2.16}
$$

Thus from (2.15) and (2.16), we deduce that g is a doubly periodic function in w with periods 1 and τ. Our next task is to show that g is an entire function of w and hence a constant (with respect to w). We show that the poles of g at $w=u$ and $w=-u$ are actually removable singularities by proving that $\lim_{w\to\pm u}(w\mp u)\left(f_2(w)+f_3(w)\right)=0$ which readily implies that $\lim_{w\to\pm u}(w\mp u)g(w)=0$. Let

$$A:=A(v,a_3,\cdots,a_m;q):=J(v)J(2v)\prod_{3\le j\le m}J(v-a_j)J(v+a_j).\tag{2.17}$$

Next, applying (2.4), (2.13) and (2.9), we see that

$$
\begin{aligned}
&\lim_{w\to u}(w-u)\left(f_2(w)+f_3(w)\right)\\
&=A\lim_{w\to u}(w-u)\Bigg\{\frac{J(v-w)J(v+w)(q)_\infty^2}{J(u-w)J(u+w)J(u-v)J(u)J(u+v)}\prod_{3\le j\le m}\frac{1}{J(u-a_j)J(u+a_j)}\\
&\quad+\frac{e^{2\pi i(v-w)}}{J(w)J(2w)}\prod_{3\le j\le m}\frac{1}{J(w-a_j)J(w+a_j)}\Bigg(\frac{1}{1-e^{2\pi i(u-w)}}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}
\end{aligned}
$$

$$\times \frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n} + \sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\Bigg)\Bigg\}$$
$$= A\frac{e^{2\pi i(v-u)}}{J(u)J(2u)}\prod_{3\le j\le m}\frac{1}{J(u-a_j)J(u+a_j)}\left(\frac{-iq^{1/8}(q)_\infty^3}{\theta'(0)}+\frac{1}{2\pi i}\right)$$
$$= 0. \tag{2.18}$$

Similarly, $\lim_{w\to -u}(w+u)\left(f_2(w)+f_3(w)\right)=0$. Now the only other possibility of a pole of g is at 0, which arises from f_1 and f_3 each having a pole at 0. Again, to show that this is a removable singularity, it suffices to show that $\lim_{w\to 0} w\left(f_1(w)+f_3(w)\right)=0$. To show this, we need Jacobi's duplication formula for theta functions [19, p. 488, Ex. 5]

$$\theta(2w)\theta_2\theta_3\theta_4 = 2\theta(w)\theta_2(w)\theta_3(w)\theta_4(w). \tag{2.19}$$

Let

$$B := B(u,v,a_3,\cdots,a_m;q) := e^{2\pi imv}\frac{J(2v)}{J(v)}\prod_{3\le j\le m}\frac{J(a_j-v)J(a_j+v)}{(J(a_j))^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi iu}q^n}. \tag{2.20}$$

Then from (2.14) and (2.20),

$$\lim_{w\to 0} w\left(f_1(w)+f_3(w)\right)$$
$$= \lim_{w\to 0} w\Bigg\{B\frac{J(w-v)J(w+v)}{(J(w))^2} + A\frac{e^{2\pi i(v-w)}}{J(w)J(2w)}\prod_{3\le j\le m}\frac{1}{J(w-a_j)J(w+a_j)}$$
$$\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\right)\Bigg\}$$
$$+\lim_{w\to 0} w\sum_{k=3}^{m} e^{2\pi i(v-a_k)}\frac{J(v)J(2v)J(v-w)J(v+w)}{J(a_k)J(2a_k)J(a_k-w)J(a_k+w)}\prod_{\substack{2<j<m+1\\ j\ne k}}\frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)}$$
$$= \lim_{w\to 0}\frac{w}{\theta(w)}\Bigg\{B\frac{\theta(w-v)\theta(w+v)}{\theta(w)} - Ae^{2\pi iv}q^{1/4}(q)_\infty^2\frac{e^{-5\pi iw}}{\theta(2w)}\prod_{3\le j\le m}\frac{1}{J(w-a_j)J(w+a_j)}$$
$$\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\right)\Bigg\}$$
$$= \frac{1}{\theta'(0)}\lim_{w\to 0}\frac{D(w)}{\theta(w)}, \tag{2.21}$$

where

$$\begin{aligned} D(w) &:= D(w, v, a_3, \cdots, a_m; q) := B\theta(w-v)\theta(w+v) - Ae^{2\pi iv}q^{1/4}(q)_\infty^2 E(w)\\ E(w) &:= E(w; u, a_3 \cdots, a_m; q) := \frac{e^{-5\pi iw}\theta(w)}{\theta(2w)} \prod_{3\le j\le m} \frac{1}{J(w-a_j)J(w+a_j)}\\ &\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n} \right). \end{aligned} \tag{2.22}$$

Now using (2.19), (2.4) and (2.13), we find that as $w \to 0$,

$$\begin{aligned} D(w) &\to -B\theta^2(v) - \frac{e^{2\pi iv}Aq^{1/4}(q)_\infty^2}{(-1)^{m-2}e^{-2\pi i(a_3+\cdots+a_m)}(J(a_3)\cdots J(a_m))^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi iu}q^n}\\ &= 0, \end{aligned} \tag{2.23}$$

which is observed by putting the expressions for B and C back in the first expression on the right side in (2.23). Thus,

$$\lim_{w\to 0} w\left(f_1(w) + f_3(w)\right) = \frac{D'(0)}{\theta'(0)^2}. \tag{2.24}$$

Now we calculate $D'(0)$.

$$D'(w) = B\left(\theta'(w-v)\theta(w+v) + \theta(w-v)\theta'(w+v)\right) - e^{2\pi iv}Aq^{1/4}(q)_\infty^2 E'(w). \tag{2.25}$$

Using (2.4) and (2.19), we have

$$\begin{aligned} E(w) &= (-1)^{m-2}q^{\frac{m-2}{4}}(q)_\infty^{2(m-2)} \frac{e^{-\pi iw(2m+1)}\theta_2\theta_3\theta_4}{2\theta_2(w)\theta_3(w)\theta_4(w)} \prod_{3\le j\le m} \frac{1}{\theta(w-a_j)\theta(w+a_j)}\\ &\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n} \right). \end{aligned} \tag{2.26}$$

Differentiating both sides with respect to w and simplifying, we obtain

$$\begin{aligned} E'(w) &= \frac{1}{2}(-1)^{m-2}q^{\frac{m-2}{4}}(q)_\infty^{2(m-2)}e^{-\pi iw(2m+1)}\theta_2\theta_3\theta_4\\ &\quad \times \left\{ -\frac{\pi i(2m+1) + \frac{\theta_2'(w)}{\theta_2(w)} + \frac{\theta_3'(w)}{\theta_3(w)} + \frac{\theta_3'(w)}{\theta_3(w)} + \sum_{3\le j\le m}\left(\frac{\theta'(w-a_j)}{\theta(w-a_j)} + \frac{\theta'(w+a_j)}{\theta(w+a_j)} \right)}{\theta_2(w)\theta_3(w)\theta_4(w) \prod_{3\le j\le m} \theta(w-a_j)\theta(w+a_j)} \right.\\ &\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n} \right)\\ &\quad \left. + \frac{1}{\theta_2(w)\theta_3(w)\theta_4(w)} \prod_{3\le j\le m} \frac{1}{\theta(w-a_j)\theta(w+a_j)} F'(w) \right\}, \end{aligned} \tag{2.27}$$

where

$$F(w):=F(w,u,m;q):=\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\right). \tag{2.28}$$

It is straightforward to see that

$$F^{'}(0)=2\pi i(2m+1)\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi iu}q^n}. \tag{2.29}$$

From (2.8), we have

$$\theta^{'}(-z)=\theta^{'}(z). \tag{2.30}$$

Then letting $w\to 0$ in (2.27), and using (2.8), (2.30), (2.29) and the fact that $\theta_k^{'}(0)=0$ for $2\le k\le 4$, we find that

$$E^{'}(0)=0. \tag{2.31}$$

Using (2.30) and (2.31) in (2.25), we finally deduce that $D^{'}(0)=0$.

With the help of (2.24), this then implies that $\lim_{w\to 0} w\,(f_1(w)+f_3(w))=0$ and thus $\lim_{w\to 0} wg(w)=0$. Thus $w=0$ is also a removable singularity, which implies that g is an doubly periodic entire function and hence a constant, say K (which may very well depend on v). Finally, since $J(0)=0$, we have

$$K=g(v)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iv(2m+1)n}}{1-e^{2\pi i(u-v)}q^n}+\frac{e^{2\pi iv(2m+1)(n+1)}}{1-e^{2\pi i(u+v)}q^n}\right).$$

This completes the proof.

3. Proof of Theorem 1.1

From [13, Eq.(4.3)], we see that

$$\begin{aligned} R_k(z,q)&=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2}(1-q^n)\left(\frac{1}{1-zq^n}+\frac{z^{-1}q^n}{1-z^{-1}q^n}\right)\\ &=\frac{z^{-1}}{(q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^{n-1}q^{n((2k-1)n+1)/2}\frac{1-q^n}{1-z^{-1}q^n}. \end{aligned} \tag{3.1}$$

Replacing z by z^{-1} in (1.11) and (3.1), we see that

$$\begin{aligned} &R_k(z,q)\\ &=\frac{z}{(q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^{n-1}q^{n((2k-1)n+1)/2}\frac{1-q^n}{1-zq^n}\\ &=\frac{z}{(q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^{n-1}q^{n((2k-1)n+1)/2}\left(1-\frac{(1-z)q^n}{1-zq^n}\right) \end{aligned}$$

$$= \frac{z}{(q)_\infty}\left(\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^{n-1}q^{n((2k-1)n+1)/2}+(1-z)\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^nq^{n((2k-1)n+3)/2}}{1-zq^n}\right)$$

$$= \frac{z}{(q)_\infty}\left(1-\sum_{n=-\infty}^{\infty}(-1)^{n}q^{n((2k-1)n+1)/2}+(1-z)\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^nq^{n((2k-1)n+3)/2}}{1-zq^n}\right)$$

$$= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty}+\frac{z}{(q)_\infty}\left(1+(1-z)\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^nq^{n((2k-1)n+3)/2}}{1-zq^n}\right)$$

$$= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty}+\frac{z}{(q)_\infty}\left(1+(1-z)\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^nq^{n((2k-1)n+3)/2}\left(\frac{z^{k-2}q^{(k-2)n}}{1-zq^n}+\frac{1-(zq^n)^{k-2}}{1-zq^n}\right)\right)$$

$$= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty}+\frac{z}{(q)_\infty}\left(1+(1-z)\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^nq^{n((2k-1)n+3)/2}\left(\frac{z^{k-2}q^{(k-2)n}}{1-zq^n}+\sum_{m=0}^{k-3}z^mq^{mn}\right)\right)$$

$$= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty}+\frac{z}{(q)_\infty}\left(1+z^{k-2}(1-z)\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^nq^{(2k-1)n(n+1)/2}}{1-zq^n}\right.$$
$$\left.+(1-z)\sum_{m=0}^{k-3}z^m\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^nq^{n((2k-1)n+2m+3)/2}\right)$$

$$= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty}+\frac{z}{(q)_\infty}\left(z^{k-2}(1-z)\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(2k-1)n(n+1)/2}}{1-zq^n}\right.$$
$$\left.+(1-z)\sum_{m=0}^{k-3}z^m\sum_{n=-\infty}^{\infty}(-1)^nq^{n((2k-1)n+2m+3)/2}\right)$$

$$= \frac{1}{(q)_\infty}\left(-z\theta_{1,2k-1}(q)+z^{k-1}(1-z)\Sigma^{(k)}(z,q)+z(1-z)\sum_{m=0}^{k-3}z^m\theta_{2m+3,2k-1}(q)\right).$$

This completes the proof of Theorem 1.1.

4. Higher order Rank-Crank-type PDEs

In this section we show how the generalized Lambert series identity (1.3) can be used to derive general Rank-Crank PDEs of the type found by Zwegers.

4.1. **The idea.** First we let $x_i = \zeta^i, 1 \le i \le m$ in (1.3) to obtain

$$Y_m(\zeta,z,q)\,(q)_\infty^2 = S_{2m+1}(\zeta,z,q)+\sum_{j=1}^{m-1}F_{j,m}(\zeta,q)\,S_{2m+1}(\zeta^{j+1},z,q)-F_{0,m}(\zeta,q)\,\Sigma^{(2m+1)}(z,q), \tag{4.1}$$

where

$$S_k(\zeta,z,q) := \sum_{n=-\infty}^{\infty}(-1)^n q^{kn(n+1)/2}\left(\frac{\zeta^{-kn}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{k(n+1)}}{1-z\zeta q^n}\right), \tag{4.2}$$

for k odd and

$$F_{0,m}(\zeta,q) := \zeta^m\frac{[\zeta^{m+1}]_\infty}{[\zeta^m]_\infty}, \tag{4.3}$$

$$F_{j,m}(\zeta,q) := \frac{\left[\zeta^{-(m-1)},\zeta^{-(m-2)},\cdots,\zeta^{-(m-j)}\right]_\infty}{\left[\zeta^{m+2},\cdots,\zeta^{m+j+1}\right]_\infty} \quad (\text{for } 1\le j\le m-1), \tag{4.4}$$

$$Y_m(\zeta,z,q) := \frac{\left[\zeta^{-(m-1)},\zeta^{-(m-2)},\cdots,\zeta^{-2},\zeta^{-1},\zeta,\zeta^2,\cdots,\zeta^m,\zeta^{m+1}\right]_\infty}{\left[z\zeta^{-m},z\zeta^{-(m-1)},\cdots,z\zeta^{-2},z\zeta^{-1},z,z\zeta,z\zeta^2,\cdots,z\zeta^{m-1},z\zeta^m\right]_\infty}. \tag{4.5}$$

The basic idea is to apply the operator D_{2m} to both sides of (4.1) where

$$D_\ell := \left(\zeta\frac{\partial}{\partial\zeta}\right)^\ell\Bigg|_{\zeta=1} = \delta_\zeta^\ell\Big|_{\zeta=1}. \tag{4.6}$$

We will also need the differential operator

$$\mathcal{H}_k^* := k\delta_z + 2k\delta_q + \delta_z^2. \tag{4.7}$$

We note that the operator $\mathcal{H}_k^*$ differs from Zwegers's $\mathcal{H}_k$ although they are similar. First we need to write the functions $\Sigma^{(k)}(z,q)$ and $S_k(z,q)$ as double series. Throughout we assume that $0<|q|<1$, $z\notin\{q^n : n\in\mathbb{Z}\}\cup\{0\}$ and $\zeta\notin\{z^{\pm1}q^n : n\in\mathbb{Z}\}\cup\{0\}$. We obtain

$$\begin{aligned}
S_k(\zeta,z,q) &= \sum_{n=0}^{\infty}(-1)^n q^{kn(n+1)/2}\left(\zeta^{-kn}\sum_{m=0}^{\infty}z^m\zeta^{-m}q^{mn}+\zeta^{k(n+1)}\sum_{m=0}^{\infty}z^m\zeta^m q^{mn}\right)\\
&\quad-\sum_{n=1}^{\infty}(-1)^n q^{kn(n-1)/2}\left(\zeta^{kn}\sum_{m=1}^{\infty}z^{-m}\zeta^m q^{mn}+\zeta^{k(-n+1)}\sum_{m=1}^{\infty}z^{-m}\zeta^{-m}q^{mn}\right)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n z^m q^{kn(n+1)/2+mn}\left(\zeta^{-kn-m}+\zeta^{k(n+1)+m}\right)\\
&\quad-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n z^{-m} q^{kn(n-1)/2+mn}\left(\zeta^{kn+m}+\zeta^{-kn+k-m}\right)
\end{aligned} \tag{4.8}$$

and

$$\Sigma^{(k)}(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n z^m q^{kn(n+1)/2+mn} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n z^{-m} q^{kn(n-1)/2+mn}. \tag{4.9}$$

We have

Theorem 4.1. *Suppose k is odd and $1\le\ell\le k-1$. Then*

$$D_\ell\, S_k(\zeta,z,q) = P_{k,\ell}(\mathcal{H}_k^*)\Sigma^{(k)}(z,q), \tag{4.10}$$

where

$$P_{k,\ell}(x) = \sum_{m=0}^{\lfloor \ell/2 \rfloor} \frac{\ell(\ell-m-1)!}{(\ell-2m)!m!} x^m k^{\ell-2m}. \tag{4.11}$$

Proof. Suppose k is odd and $1 \le \ell \le k-1$. First we prove that

$$P_{k,\ell}(x) = (\tfrac{1}{2}k - \tfrac{1}{2}\sqrt{k^2+4x})^\ell + (\tfrac{1}{2}k + \tfrac{1}{2}\sqrt{k^2+4x})^\ell. \tag{4.12}$$

We have

$$(\tfrac{1}{2}k - \tfrac{1}{2}\sqrt{k^2+4x})^\ell + (\tfrac{1}{2}k + \tfrac{1}{2}\sqrt{k^2+4x})^\ell = \sum_{j=0}^{\lfloor \ell/2 \rfloor} \binom{\ell}{2j} k^{\ell-2j} 2^{1-\ell} (k^2+4x)^j$$

$$= \sum_{j=0}^{\lfloor \ell/2 \rfloor} \sum_{m=0}^{j} \binom{\ell}{2j}\binom{j}{m} x^m k^{\ell-2m} 2^{2m-\ell+1} = \sum_{m=0}^{\lfloor \ell/2 \rfloor} \left(\sum_{j=m}^{\lfloor \ell/2 \rfloor} \binom{\ell}{2j}\binom{j}{m} \right) x^m k^{\ell-2m} 2^{2m-\ell+1}.$$

The result (4.12) now follows from the binomial coefficient identity

$$\sum_{j=m}^{\lfloor \ell/2 \rfloor} \binom{\ell}{2j}\binom{j}{m} = 2^{\ell-2m-1} \frac{\ell(\ell-m-1)!}{(\ell-2m)!m!}, \tag{4.13}$$

which we leave as an exercise.

We observe that if $x = km + m^2 + k^2 n(n+1) + 2mnk$ then

$$\begin{aligned} k^2+4x &= (k+2m+2kn)^2, \\ \tfrac{1}{2}k - \tfrac{1}{2}\sqrt{k^2+4x} &= -kn-m, \\ \tfrac{1}{2}k + \tfrac{1}{2}\sqrt{k^2+4x} &= k(n+1)+m, \end{aligned}$$

and we see that

$$D_\ell\left(\zeta^{-kn-m} + \zeta^{k(n+1)+m}\right) = (-kn-m)^\ell + (k(n+1)+m)^\ell = P_{k,\ell}(km+m^2+k^2n(n+1)+2mnk).$$

Similarly we find that

$$D_\ell\left(\zeta^{kn+m} + \zeta^{-kn+k-m}\right) = (kn+m)^\ell + (-kn+k-m)^\ell = P_{k,\ell}(-km+m^2+k^2n(n-1)+2mnk).$$

We note that

$$\begin{aligned} \mathcal{H}_k^*\left(q^{kn(n+1)/2+mn} z^m\right) &= (km+m^2+k^2n(n+1)+2mnk)\left(q^{kn(n+1)/2+mn} z^m\right), \\ \mathcal{H}_k^*\left(q^{kn(n-1)/2+mn} z^{-m}\right) &= (-km+m^2+k^2n(n-1)+2mnk)\left(q^{kn(n-1)/2+mn} z^{-m}\right). \end{aligned}$$

Thus

$$D_\ell\left(q^{kn(n+1)/2+mn} z^m (\zeta^{-kn-m} + \zeta^{kn+k+m})\right) = P_{k,\ell}(\mathcal{H}_k^*)\left(q^{kn(n+1)/2+mn} z^m\right),$$

and

$$D_\ell\left(q^{kn(n+1)/2+mn} z^m (\zeta^{kn+m} + \zeta^{-kn+k-m})\right) = P_{k,\ell}(\mathcal{H}_k^*)\left(q^{kn(n-1)/2+mn} z^{-m}\right).$$

The result (4.10) follows from equations (4.8) and (4.9). ∎

Next we calculate D_{2m} of each term in (4.1).

4.2. **The term $Y_m(\zeta,z,q)$.** It is clear that the function $Y_m(\zeta,z,q)$ has a zero of order $2m$ at $\zeta=1$. It is well-known that

$$D_{2m}(f(\zeta)) = \sum_{i=1}^{2m} S(2m,j) f^{(j)}(1),$$

where the numbers $S(2m,j)$ are Stirling numbers of the second kind. Since $S(2m,2m)=1$ it follows that

$$D_{2m}\left(Y_m(\zeta,z,q)\right)) = Y_m^{(2m)}(1,z,q) = (-1)^{m-1}(2m)!\,(m+1)!\,(m-1)!\,[C^*(z,q)]^{2m+1}(q)_\infty^{2m-1} \tag{4.14}$$

by an easy calculation.

4.3. **The term $F_{0,m}(\zeta,q)$.** By logarithmic differentiation we have

$$\delta_\zeta F_{0,m}(\zeta,q) = L_{0,m}(\zeta,q)\,F_{0,m}(\zeta,q). \tag{4.15}$$

where

$$\begin{aligned} L_{0,m}(\zeta,q) &= K_{0,m}(\zeta) - (m+1)\sum_{i=1}^{\infty}\left(\frac{\zeta^{m+1}q^i}{1-\zeta^{m+1}q^i} - \frac{\zeta^{-m-1}q^i}{1-\zeta^{-m-1}q^i}\right) \\ &\qquad + m\sum_{i=1}^{\infty}\left(\frac{\zeta^{m}q^i}{1-\zeta^{m}q^i} - \frac{\zeta^{-m}q^i}{1-\zeta^{-m}q^i}\right) \\ &= K_{0,m}(\zeta) - (m+1)\sum_{i,n\ge1}(\zeta^{m+1}q^i)^n - (\zeta^{-m-1}q^i)^n + m\sum_{i,n\ge1}(\zeta^{m}q^i)^n - (\zeta^{-m}q^i)^n \end{aligned} \tag{4.16}$$

$$K_{0,m}(\zeta) = m + J_m(\zeta) - J_{m-1}(\zeta), \tag{4.17}$$

$$J_m(\zeta) = \frac{\displaystyle\sum_{n=1}^{m} n\zeta^n}{\displaystyle\sum_{n=0}^{m}\zeta^n}. \tag{4.18}$$

For any positive integer k we define

$$G_{2k} := G_{2k}(q) := \frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n} = -\frac{B_{2k}}{4k} + \Phi_{2k-1}(q), \tag{4.19}$$

where B_{2n} is the $(2n)$-th Bernoulli number, and

$$\Phi_{2k-1} := \Phi_{2k-1}(q) := \sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n. \tag{4.20}$$

The function G_{2k} is a normalized Eisenstein series. For $k>1$ it is an entire modular form of weight $2k$. We need the following

Lemma 4.2. *If m and a are positive integers, then*

$$D_a(J_m(\zeta)) = \frac{B_{a+1}}{a+1}\left((m+1)^{a+1}-1\right).$$

Proof. Suppose m and a are positive integers. It is well-known that

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}. \tag{4.21}$$

By taking the logarithmic derivative of $(\zeta^{m+1} - 1)/(\zeta - 1)$ we find that

$$J_m(\zeta) = m + (m+1)\frac{1}{\zeta^{m+1} - 1} - \frac{1}{\zeta - 1}.$$

Hence by (4.21) we have

$$J_m(e^x) = m + \sum_{k=0}^{\infty} \frac{B_{k+1}}{(k+1)!} \left((m+1)^{k+1} - 1\right) x^k.$$

The result now follows since

$$D_a(J_m(\zeta)) = \left(\frac{d}{dx}\right)^a J_m(e^x)\bigg|_{x=0}.$$

□

Corollary 4.3. *Suppose a, m are integers $a \geq 0$ and $m \geq 1$. Then*

$$D_a(L_{0,m}(\zeta, q)) = \begin{cases} 2(m^{a+1} - (m+1)^{a+1})G_{a+1}(q) & \text{if } a \text{ is odd,} \\ m + \frac{1}{2} & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

Proof. The proof of (4.22) when $a = 0$ is straightforward. Suppose a is even and positive. Then by Lemma 4.2

$$\begin{aligned} D_a(L_{0,m}(\zeta, q)) &= D_a(K_{0,m}(\zeta)) \\ &= \frac{B_{a+1}}{a+1}(-m^{a+1} + (m+1)^{a+1}) \\ &= 0, \end{aligned}$$

since the Bernoulli numbers B_k are zero when $k > 1$ is odd. Now suppose a is odd. Then again by Lemma 4.2

$$\begin{aligned} D_a(L_{0,m}(\zeta, q)) &= D_a(K_{0,m}(\zeta)) + 2(m^{a+1} - (m+1)^{a+1})\Phi_a(q) \\ &= \frac{B_{a+1}}{a+1}(-m^{a+1} + (m+1)^{a+1}) + 2(m^{a+1} - (m+1)^{a+1})\Phi_a(q) \\ &= 2(m^{a+1} - (m+1)^{a+1})G_{a+1}(q). \end{aligned}$$

□

By applying D_{a-1} to both sides of (4.15) and using (4.22) we obtain the following recurrence

$$\begin{aligned} D_a(F_{0,m}(\zeta, q)) &= (m + \tfrac{1}{2})D_{a-1}(F_{0,m}(\zeta, q)) \\ &\quad + \sum_{i=1}^{\lfloor a/2 \rfloor} 2\binom{a-1}{2i-1}(m^{2i} - (m+1)^{2i})G_{2i}(q)D_{a-2i}(F_{0,m}(\zeta, q)). \end{aligned} \tag{4.23}$$

This together with the initial value

$$D_0(F_{0,m}(\zeta,q)) = F_{0,m}(1,q) = \tfrac{m+1}{m} \tag{4.24}$$

uniquely determines the coefficients $D_a(F_{0,m}(\zeta,q))$. We compute some examples

$$\begin{aligned}
D_0(F_{0,2}) &= \frac{3}{2},\\
D_1(F_{0,2}) &= \frac{15}{4},\\
D_2(F_{0,2}) &= \frac{75}{8} - 15G_2 = 10 - 15\Phi_1,\\
D_3(F_{0,2}) &= \frac{365}{16} - \frac{225}{2}G_2 = \frac{225}{8} - \frac{225}{2}\Phi_1,\\
D_4(F_{0,2}) &= \frac{1875}{32} - \frac{1125}{2}G_2 + 450G_2^2 - 195G_4 = 82 - 600\Phi_1 + 450\Phi_1^2 - 195\Phi_3.
\end{aligned}$$

4.4. **The terms $F_{j,m}(\zeta,q)$ $(1 \le j \le m-1)$.** Suppose $1 \le j \le m-1$. We may obtain a similar recurrence for $D_a(F_{j,m}(\zeta,q))$. This time we find that

$$\delta_\zeta F_{j,m}(\zeta,q) = L_{j,m}(\zeta,q)\, F_{j,m}(\zeta,q), \tag{4.25}$$

for some function $L_{j,m}(\zeta,q)$ that satisfies

$$D_a(L_{j,m}(\zeta,q)) = \begin{cases} 2\sum_{i=1}^{j}\left((m+i+1)^{a+1} - (m-i)^{a+1}\right) G_{a+1}(q) & \text{if } a \text{ is odd},\\ -j\left(m+\frac{1}{2}\right) & \text{if } a = 0,\\ 0 & \text{otherwise.}\end{cases} \tag{4.26}$$

The proof of (4.26) is analogous to the proof of Corollary 4.3. By applying D_{a-1} to both sides of (4.25) and using (4.26) we obtain the following recurrence

$$\begin{aligned}
D_a(F_{j,m}(\zeta,q)) &= -j(m+\tfrac{1}{2})D_{a-1}(F_{j,m}(\zeta,q))\\
&\quad + \sum_{i=1}^{\lfloor a/2\rfloor}\sum_{k=1}^{j} 2\binom{a-1}{2i-1}((m+k+1)^{2i} - (m-k)^{2i})G_{2i}(q)D_{a-2i}(F_{j,m}(\zeta,q)).
\end{aligned} \tag{4.27}$$

This together with the initial value

$$D_0(F_{j,m}(\zeta,q)) = F_{j,m}(1,q) = (-1)^j \prod_{i=1}^{j} \frac{m-i}{m+i+1} \tag{4.28}$$

uniquely determines the coefficients $D_a(F_{j,m}(\zeta,q))$. We compute some examples

$$\begin{aligned}
D_0(F_{1,2}) &= -\frac{1}{4},\\
D_1(F_{1,2}) &= \frac{5}{8},\\
D_2(F_{1,2}) &= -\frac{25}{16} - \frac{15}{2}G_2 = -\frac{5}{4} - \frac{15}{2}\Phi_1,\\
D_3(F_{1,2}) &= \frac{125}{32} + \frac{225}{4}G_2 = \frac{25}{16} + \frac{225}{4}\Phi_1,\\
D_4(F_{1,2}) &= -\frac{625}{64} - \frac{1125}{4}G_2 - 675G_2^2 - \frac{255}{2}G_4 = \frac{1}{4} - 225\Phi_1 - 675\Phi_1^2 - \frac{255}{2}\Phi_3.
\end{aligned}$$

4.5. **The terms $S_{2m+1}(\zeta^{j+1},z,q)$ $(1 \le j \le m-1)$.** Again suppose that $1 \le j \le m-1$. Consider the operator T_j that operates on a function $f(\zeta)$ by $T_j(f(\zeta)) = f(\zeta^j)$. Then

$$\delta_\zeta \circ T_j = j\ (T_j \circ \delta_\zeta)\,. \tag{4.29}$$

A simple induction argument gives

$$\delta_\zeta^a \circ T_j = j^a\ \left(T_j \circ \delta_\zeta^a\right), \tag{4.30}$$

and

$$D_a \circ T_j = j^a\, D_a. \tag{4.31}$$

Thus by (4.10) we have

$$D_\ell S_{2m+1}(\zeta^{j+1},z,q) = (j+1)^\ell P_{2m+1,\ell}(\mathcal{H}_{2m+1}^*)\Sigma^{(2m+1)}(z,q). \tag{4.32}$$

4.6. **The main theorem.** We are now ready to derive our main theorem. Applying D_{2m} to both sides of (4.1) and using (4.10), (4.14), (4.23), (4.24), (4.27), (4.28), and (4.32) we have

Theorem 4.4.

$$\begin{aligned}
&(-1)^{m+1}(2m)!\,(m+1)!\,(m-1)![C^*(z,q)]^{2m+1}(q)_\infty^{2m+1}\\
&= \Bigg(P_{2m+1,2m}(\mathcal{H}_{2m+1}^*)\\
&\quad + \sum_{j=1}^{m-1}\sum_{a=0}^{2m}(j+1)^{2m-a}\binom{2m}{a}D_a(F_{j,m}(\zeta,q))P_{2m+1,2m-a}(\mathcal{H}_{2m+1}^*)\\
&\quad - D_{2m}(F_{0,m}(\zeta,q))\Bigg)\Sigma^{(2m+1)}(z,q)
\end{aligned} \tag{4.33}$$

where the coefficient functions $D_a(F_{j,m})(\zeta,q))$ $(0 \le j \le m-1)$ are given recursively by (4.23) and (4.27), and their initial values (4.24) and (4.28).

For $n \ge 0$ let $\mathcal{V}_n$ be the $\mathbb{Q}$-vector space spanned by the monomials $\Phi_1^a\Phi_3^b\Phi_5^c$ with $a+2b+3c=n$. We Define

$$\mathcal{W}_n = \sum_{k=0}^{n}\mathcal{V}_n; \tag{4.34}$$

i.e., $\mathcal{W}_n$ is the $\mathbb{Q}$-vector space spanned by the monomials $\Phi_1^a\Phi_3^b\Phi_5^c$ with $0 \le a+2b+3c \le n$. We call $\mathcal{W}_n$ the space of *quasi-modular forms* of weight less than or equal to $2n$. This agrees with the definition in [2, p.355] except this time we allow monomials of weight 0.

Corollary 4.5. *Suppose $m \ge 1$. Then there exist quasi-modular forms $f_j \in \mathcal{W}_j$ for $1 \le j \le m$ such that*

$$\left(\mathcal{H}_{2m+1}^{*\,m} + \sum_{k=0}^{m-1} f_{m-k}\,\mathcal{H}_{2m+1}^{*\,k}\right)\Sigma^{(2m+1)}(z,q) = (2m)!\,[C^*(z,q)]^{2m+1}(q)_\infty^{2m+1}. \tag{4.35}$$

Proof. Suppose $m \ge 1$. It is well-known that

$$\Phi_{2n-1} \in \mathcal{W}_n.$$

See [2, Eq.(3.25)]. Equation (4.19), the recurrence (4.27) and a simple induction argument imply that

$$D_a(F_{j,m}(\zeta,q)) \in \mathcal{W}_{\lfloor a/2 \rfloor},$$

for $1 \le j \le m-1$. Similarly using (4.22) and (4.23) we find that

$$D_{2m}(F_{0,m}(\zeta,q)) \in \mathcal{W}_m. \tag{4.36}$$

Now we calculate the coefficient of $\mathcal{H}_{2m+1}^{*\,k}$ in the right side of (4.33). The degree of the polynomial $P_{2m+1,2m-a}(x)$ is $\lfloor (2m-a)/2 \rfloor$. Assuming $k \le \lfloor (2m-a)/2 \rfloor$ we have $2k \le 2m-a$ and $\lfloor a/2 \rfloor \le m-k$, and in this case $D_a(F_{j,m}(\zeta,q))$ is in $\mathcal{W}_{m-k}$. This together with (4.36) implies that the coefficient of $\mathcal{H}_{2m+1}^{*\,k}$ is in $\mathcal{W}_{m-k}$ for $0 \le k \le m$. The coefficient of $\mathcal{H}_{2m+1}^{*\,m}$ is

$$\begin{aligned} f_0 &= 2 + 2\sum_{j=1}^{m-1}(-1)^j(j+1)^{2m}\prod_{k=1}^{j}\frac{(m-k)}{m+k+1} \\ &= 2\sum_{j=0}^{m-1}(-1)^j(j+1)^{2m}\frac{(m-1)!(m+1)!}{(m-j-1)!(m+j+1)!} \\ &= \frac{2(m-1)!(m+1)!}{(2m)!}\sum_{j=0}^{m-1}(-1)^j(j+1)^{2m}\binom{2m}{m-j-1}. \end{aligned} \tag{4.37}$$

We show that

$$f_0 = (-1)^{m+1}\,(m+1)!\,(m-1)!. \tag{4.38}$$

In view of (4.37) this is equivalent to showing that

$$\frac{2}{(2m)!}\sum_{j=0}^{m-1}(-1)^{m+j+1}(j+1)^{2m}\binom{2m}{m-j-1} = 1, \tag{4.39}$$

which we can rewrite as

$$2\sum_{j=0}^{m-1}(-1)^j(m-j)^{2m}\binom{2m}{j} = (2m)!, \tag{4.40}$$

by replacing j by $m-j-1$ in the sum.

Since

$$\sum_{j=0}^{m-1}(-1)^j(m-j)^{2m}\binom{2m}{j} = \sum_{j=m+1}^{2m}(-1)^j(m-j)^{2m}\binom{2m}{j},$$

it suffices to prove

$$\sum_{j=0}^{2m}(-1)^j(m-j)^{2m}\binom{2m}{j} = (2m)!. \tag{4.41}$$

Beginning with the elementary identity

$$\sum_{j=0}^{2m}\binom{2m}{j}x^j y^{2m-j} = (x+y)^{2m},$$

setting $x = -\frac{1}{\sqrt{\zeta}}$ and $y = \sqrt{\zeta}$, we obtain

$$\sum_{j=0}^{2m}\binom{2m}{j}(-1)^j\zeta^{m-j} = \zeta^{-m}(\zeta-1)^{2m}. \tag{4.42}$$

We apply D_{2m} to both sides of (4.42) and argue as in Section 4.2 to obtain (4.41) which completes the proof of (4.38). The final result (4.35) follows from (4.33) by dividing both sides by f_0 and using (4.38). □

4.7. **Some examples.** We illustrate Theorem 4.4 and Corollary 4.5 with some examples. We show details of the calculations for the cases $m = 1, 2$. In cases $m = 3, 4$ we give the quasi-modular forms f_j $(1 \le j \le m)$ in Corollary 4.5, in terms of the functions $\Phi_{2k-1}(q)$ rather than the $G_{2k}(q)$.

Example $m = 1$.

$$\begin{aligned}
4\,[C^*(z,q)]^3(q)_\infty^3 &= \left(9 + 2\,\mathcal{H}_3 - D_2(F_{0,1}(\zeta,q))\right)\Sigma^{(3)}(z,q)\\
&= \left(9 + 2\,\mathcal{H}_3 - (5 - 12\,\Phi_1)\right)\Sigma^{(3)}(z,q)\\
&= \left(2\,\mathcal{H}_3 + 4 + 12\,\Phi_1\right)\Sigma^{(3)}(z,q),
\end{aligned}$$

and

$$\left(\mathcal{H}_3 + 2 + 6\,\Phi_1\right)\Sigma^{(3)}(z,q) = 2\,[C^*(z,q)]^3(q)_\infty^3.$$

This identity implies the Rank-Crank PDE (1.7) as in [2, Section 2].

Example $m = 2$.

$$
\begin{aligned}
&- 144\,[C^*(z,q)]^5 (q)_\infty^5 \\
&= \Bigg(625 + 100\,\mathcal{H}^*{}_5 + 2\,\mathcal{H}^*{}_5^2 + 16\,D_0(F_{1,2}(\zeta,q))\,(625 + 100\,\mathcal{H}^*{}_5 + 2\,\mathcal{H}^*{}_5^2) \\
&+ 32\,D_1(F_{1,2}(\zeta,q))\,(125 + 15\,\mathcal{H}^*{}_5) + 24\,D_2(F_{1,2}(\zeta,q))\,(25 + 2\,\mathcal{H}^*{}_5) + 40\,D_3(F_{1,2}(\zeta,q)) \\
&+ 2\,D_4(F_{1,2}(\zeta,q)) - D_4(F_{0,2}(\zeta,q))\Bigg)\Sigma^{(5)}(z,q) \\
&= \Bigg(625 + 100\,\mathcal{H}^*{}_5 + 2\,\mathcal{H}^*{}_5^2 - 4\,(625 + 100\,\mathcal{H}^*{}_5 + 2\,\mathcal{H}^*{}_5^2) \\
&+ 20\,(125 + 15\,\mathcal{H}^*{}_5) - (30 + 180\,\Phi_1)(25 + 2\,\mathcal{H}^*{}_5) + (\frac{125}{2} + 2250\,\Phi_1) \\
&+ (\frac{1}{2} - 450\,\Phi_1 - 1350\,\Phi_1^2 - 255\,\Phi_3) + (-82 + 600\,\Phi_1 - 450\Phi_1^2 + 195\,\Phi_3)\Bigg)\Sigma^{(5)}(z,q),
\end{aligned}
$$

and

$$
\Bigg(\mathcal{H}^*{}_5^2 + (60\Phi_1 + 10)\,\mathcal{H}^*{}_5 + 300\,\Phi_1^2 + 10\,\Phi_3 + 350\,\Phi_1 + 24\Bigg)\Sigma^{(5)}(z,q) = 24\,[C^*(z,q)]^5 (q)_\infty^5. \tag{4.43}
$$

In this case of Corollary 4.5 we see that

$$
\begin{aligned}
f_1 &= 60\Phi_1 + 10, \\
f_2 &= 300\,\Phi_1^2 + 10\,\Phi_3 + 350\,\Phi_1 + 24.
\end{aligned}
$$

We show how this identity implies (1.18). We need the results

$$
\delta_q((q)_\infty) = -\Phi_1\,(q)_\infty, \qquad \delta_q(\Phi_1) = \tfrac{1}{6}\Phi_1 - 2\Phi_1^2 + \tfrac{5}{6}\Phi_3.
$$

This implies that

$$
\mathcal{H}_5^*(\Sigma^*(z,q)) = \mathcal{H}_5^*((q)_\infty^3 G^{(5)}(z,q)) = (q)_\infty^3(\mathcal{H}_5^* - 30\Phi_1)G^{(5)}(z,q), \tag{4.44}
$$

and

$$
\mathcal{H}_5^{*\,2}(\Sigma^*(z,q)) = \mathcal{H}_5^{*\,2}((q)_\infty^3 G^{(5)}(z,q)) = (q)_\infty^3(\mathcal{H}_5^{*\,2} - 60\Phi_1\mathcal{H}_5^* - 50\Phi_1 + 1500\Phi_1^2 - 250\Phi_3)G^{(5)}(z,q). \tag{4.45}
$$

Substituting (4.44), (4.45) into (4.43) we find

$$
(\mathcal{H}_5^{*\,2} + 10\mathcal{H}_5^* + 24 - 240\Phi_3)G^{(5)}(z,q) = 24\,[C^*(z,q)]^5 (q)_\infty^5,
$$

which simplifies to (1.18) since

$$
\mathbf{H}_* = \mathcal{H}_5^* + 5.
$$

Example $m = 3$. We find that

$$f_1 = 210\,\Phi_1 + 28$$
$$f_2 = 210\,\Phi_3 + 8820\,\Phi_1^2 + 252 + 4410\,\Phi_1$$
$$f_3 = 41160\,\Phi_1^3 + 2450\,\Phi_3 + 14\,\Phi_5 + 720 + 22736\,\Phi_1 + 2940\,\Phi_3\,\Phi_1 + 102900\,\Phi_1^2$$

Example $m = 4$. We find that

$$f_1 = 504\,\Phi_1 + 60$$
$$f_2 = 1260\,\Phi_3 + 24948\,\Phi_1 + 1308 + 68040\,\Phi_1^2$$
$$f_3 = 136080\,\Phi_3\,\Phi_1 + 504\,\Phi_5 + 45360\,\Phi_3 + 2449440\,\Phi_1^2 + 403704\,\Phi_1 + 2449440\,\Phi_1^3 + 12176$$
$$\begin{aligned} f_4 = {} & 40320 + 2126232\,\Phi_1 + 404082\,\Phi_3 + 9828\,\Phi_5 + 2653560\,\Phi_3\,\Phi_1 + 21820428\,\Phi_1^2 \\ & + 9072\,\Phi_1\,\Phi_5 + 47764080\,\Phi_1^3 + 18\,\Phi_7 + 1224720\,\Phi_3\,\Phi_1^2 + 11340\,\Phi_3^2 + 11022480\,\Phi_1^4. \end{aligned}$$

5. Concluding remarks

The main goal of this paper was to show how the generalized Lambert series identity (1.3) leads to the higher level Rank-Crank-type PDEs of Zwegers. The first author's proof [8] of (1.3) only involves a partial fraction argument and this together with the proof in Section 4 gives an elementary q-series proof of these higher level Rank-Crank-type PDEs. The elliptic function proof of (1.3) in Section 2 is independent of the other sections. Our form of Zwegers's result (1.23) was given above in (4.35). In our form the coefficients are quasimodular forms rather than holomorphic modular forms. The quasimodular function E_2 occurs in Zwegers's result as part of the definition of his operator $\mathcal{H}_k$. Our coefficient functions are given recursively. It would be interesting to find explicit expressions for the coefficients and to derive the form of Zwegers's result by our method. The coefficients in the 4^{th} order PDE (1.18) only involve the holomorphic modular form E_4, and the differential operator $\mathbf{H}_*$ does not involve the quasimodular E_2. It would be interesting to determine whether there is a renormalization of higher order Rank-Crank-type PDEs which only involve holomorphic modular forms, either as coefficients or in the definition of the differential operator. Bringmann, Lovejoy and Osburn [5], [6] found Rank-Crank-type PDEs for overpartitions. Bringmann and Zwegers [7] showed how these results fit into the framework of non-holomorphic Jacobi forms and found an infinite family of these PDEs. However these PDEs only involve Appell functions of level 1 or 3. It would be interesting to determine whether the methods of this paper could be extended to find PDEs for higher level analogues.

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